

Cosmological model with local symmetry of very special relativity and constraints on it from supernovae

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Based on Cohen & Glashow's very special relativity [A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. **97** (2006) 021601], we propose an anisotropic modification to the Friedmann-Robertson-Walker (FRW) line element. An arbitrarily oriented 1-form is introduced and the FRW spacetime becomes of the Randers-Finsler type. The 1-form picks out a privileged axis in the universe. Thus, the cosmological redshift as well as the Hubble diagram of the type Ia supernovae (SNe Ia) becomes anisotropic. By directly analyzing the Union2.1 compilation, we obtain the privileged axis pointing to $(l, b) = (242^\circ \pm 44^\circ, -42^\circ \pm 23^\circ)$ (68% C.L.). This privileged axis is close to those obtained by comparing the best-fit Hubble diagrams in pairs of hemispheres.

PACS numbers: 98.80.Es, 98.80.Jk

1. INTRODUCTION

The cosmological principle is one of foundations of the standard cosmological model, i.e., the Λ CDM model [1]. It says that the universe is statistically homogeneous and isotropic at large scale. The Λ CDM model is well consistent with present cosmological observations, such as the Wilkinson Microwave Anisotropy Probe (WMAP) [2] and the Sloan Digital Sky Survey (SDSS) [3], etc. However, there exist challenges for the standard cosmological model (see review in Ref. [4]), such as the large-scale cosmic flows [5], the alignment of low multipoles in the CMB spectra [6–8], the large-scale alignment of the quasar polarization vectors [9]. One of their resolutions refers to a privileged axis at large scale in the universe [10].

The type Ia supernovae (SNe Ia) have been used to search for the possible anisotropy of the universe [11–22], since they were employed to discover the cosmic acceleration [23, 24]. Especially, Antoniou & Perivolaropoulos [10] used the hemisphere comparison method to analyze the Union2 data [25] and found a direction $(l, b) = (309^\circ_{-3^\circ}^{+23^\circ}, 18^\circ_{-10^\circ}^{+11^\circ})$ for the maximum accelerating expansion of the universe. Similarly, Cai & Tuo [26] found a preferred direction $(l, b) = (314^\circ_{-13^\circ}^{+20^\circ}, 28^\circ_{-33^\circ}^{+11^\circ})$ for the cosmological deceleration parameter. Most recently, Kalus *et al.* [27] also found that the highest expansion rate of the universe towards the direction $(l, b) \approx (-35^\circ, -19^\circ)$ (95 % C.L.).

A privileged axis, as is mentioned above, may account

for the anisotropic phenomenologies of the observational astrophysics and cosmology. Actually, the issue of privileged axis has been studied extensively in the very special relativity (VSR) [28]. The Finslerian spacetime structure $d\tau = (\eta_{\mu\nu} dx^\mu dx^\nu)^{\frac{1-b}{2}} (n_\sigma dx^\sigma)^b$ was proved to be invariant under the $DISIM_b(2)$ group [29]. There is a preferred axis n_σ in the above line element. It could characterize the anisotropy of the flat spacetime, which leads to the Lorentz invariance violation (LIV). Locally, the Randers metric was proved to possess the symmetry of the group $TE(2)$ [30, 31]. The group $TE(2)$ is a semi-product of $T(4)$ and $E(2)$ [32]. The group $E(2)$ denotes a proper subgroup of the Lorentz group with three generators [28]. Therefore, Randers spacetime could possess local symmetry of the generic VSR.

In this paper, we propose a Randers line element (structure) [33] with local symmetry of the generic VSR to estimate the possible anisotropy of the Hubble diagram of SNe Ia. A modified cosmological redshift formula is presented. The redshift is direction dependent and refers a privileged axis. This may imply that the universe undergoes an anisotropic expansion. Next, we show a modified luminosity-distance vs. redshift relation for the SNe Ia. It is certainly anisotropic. The Union2.1 compilation [34] is used to constrain the direction of the privileged axis. The rest of the paper is arranged as follows. In section 2, the anisotropic Hubble diagram is showed in a Randers spacetime. In section 3, we investigate the Union2.1 dataset of the supernovae to constrain the level of the anisotropy of the universe. Conclusions and discussions are listed in section 4.

2. THE ANISOTROPIC HUBBLE DIAGRAM

In the Λ CDM model, the cosmic spacetime is described by the spatially flat Friedmann-Robertson-

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Walker (FRW) line element [1]

$$\overline{d\tau} = \sqrt{dt^2 - a^2(t) [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)]} , \quad (1)$$

where $a(t)$ denotes the scale factor of the universe at the time t . The cosmological redshift \bar{z} is given by

$$1 + \bar{z}(t) = \frac{a(t_0)}{a(t)} \equiv \frac{1}{a(t)} , \quad (2)$$

where we set $a(t_0) \equiv 1$ for today. The redshift describes the expansion rate of the universe between t and t_0 . The overline denotes physical objects in the FRW-Riemannian spacetime.

The FRW line element should be modified when there exists a privileged axis in the universe. We postulate that an extra closed 1-form is added into the FRW structure. The 1-form singles out a privileged axis in the universe. The spacetime structure becomes

$$d\tau \equiv \overline{d\tau} + \tilde{b}_\mu(x) dx^\mu , \quad (3)$$

where the 1-form is closed, i.e., $\tilde{b}_{\nu|\lambda} - \tilde{b}_{\lambda|\nu} = 0$. Here $\tilde{b}_{\mu|\nu}$ denotes the covariant derivative of $\tilde{b}_\mu(x)$ with respect to the FRW metric [35]. Actually, the spacetime structure (3) belongs to Randers type [33]. This is a class of Finsler spacetime [35]. The 1-form could be viewed as an arbitrarily oriented electromagnetic 4-potential in the universe. It may be the relic of a primordial magnetic field at large scales [36–39].

In the FRW-Randers spacetime, the cosmological redshift z could be derived via resolving the Finslerian geodesic equations, which are given by [40]

$$\frac{d^2 x^0}{d\tau^2} + \delta_{ij} \dot{a} a \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} + \frac{dx^0}{d\tau} f \left(x, \frac{dx}{d\tau} \right) = 0 , \quad (4)$$

$$\frac{d^2 x^i}{d\tau^2} + 2\delta_j^i \dot{a} \frac{dx^0}{d\tau} \frac{dx^j}{d\tau} + \frac{dx^i}{d\tau} f \left(x, \frac{dx}{d\tau} \right) = 0 , \quad (5)$$

where $f(x, \frac{dx}{d\tau}) \equiv \tilde{b}_{\nu|\lambda} \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} / F$ and the dots over a denote $\frac{d}{d\tau}$. The above geodesic equations have solutions as $a \frac{dx^0}{d\tau} \propto J_1$ and $a^2 \frac{dx^i}{d\tau} \propto J_1$, where $J_1 = 1 - \tilde{b}_\mu \hat{p}^\mu$ [40]. Thus, the redshift z could be written as

$$1 + z(t, \hat{p}) = \frac{1}{a(t)} \left(1 - \tilde{b}_\mu \hat{p}^\mu \right) , \quad (6)$$

where the unit 4-vector \hat{p} denotes a light-like direction towards each SN Ia. It is anisotropic since there is a privileged axis $\tilde{b}_\mu(x)$ in the above formula.

When the privileged axis is a spatial 3-vector, the modified redshift z could be rewritten as

$$1 + z(t, \hat{\mathbf{p}}) = \frac{1}{a(t)} [1 - D(\hat{\mathbf{n}} \cdot \hat{\mathbf{p}})] , \quad (7)$$

where $\hat{\mathbf{n}}$ is the unit 3-vector for the spatial components of \tilde{b} , and D denotes the magnitude which is smaller than

one. To simplify our following discussions, we choose the spatial 3-vector $\hat{\mathbf{n}}$ as the third spatial axis. Thus, the redshift z becomes

$$1 + z(t, \cos \theta) = \frac{1}{a} [1 - D \cos \theta] , \quad (8)$$

where θ denotes the angle between $\hat{\mathbf{n}}$ and $\hat{\mathbf{p}}$. This formula shows clearly that the cosmological redshift is direction dependent and anisotropic. Thus, the universe undergoes an anisotropic expansion, which takes a dipole form.

The Hubble diagram would be anisotropic and have a corresponded privileged axis which exists in the Randers spacetime. We will discuss this proposition in the following paragraphs. For the null geodesic, the spacetime line element vanishes, i.e., $d\tau = 0$. In the FRW-Randers spacetime, the Randers structure (3) should vanishes. Thus, we obtain

$$dt^2 - a^2 dr^2 = (-D \cos \theta dr)^2 , \quad (9)$$

where we use the polar coordinates and $\hat{\mathbf{n}}$ is set again as the third spatial axis. Here the angular coordinates are discarded since they are irrelative to the distance-redshift relation. Finally, the above equation could be simplified as

$$dt = \sqrt{a^2 + D^2 \cos^2 \theta} dr . \quad (10)$$

The privileged axis emerges again, which affects the propagation of the cosmic light. This result is different from that of $dt = a(t)dr$ in the Λ CDM model. The modification is quadrupolar in Eq. (10). Obviously, the cosmic light would propagate with different speeds in different directions in the space. However, the quadrupolar effect is a second-order effect. It just affects the propagation of the cosmic light slightly.

In the observational universe, the luminosity distance d_L of a SN Ia is given by [41]

$$d_L \equiv (1 + z)r , \quad (11)$$

where r denotes the comoving distance between us and the SN Ia. We could obtain the comoving distance r by integrating the equation (10). Therefore, the luminosity-distance vs. redshift relation can be obtained as

$$d_L = (1 + z) \int_{t_0}^{t(z)} \mathcal{B}(a, \theta) dt , \quad (12)$$

where

$$\mathcal{B}(a, \theta) \equiv (a^2 + D^2 \cos^2 \theta)^{-1/2} . \quad (13)$$

The Eq. (12) could be rewritten as

$$\frac{d_L}{1 + z} = \int_1^{(1+z)^{-1}} \mathcal{B}(a, \theta) \frac{da}{a H_0 \sqrt{\Omega_m a^{-3} + \Omega_\Lambda}} , \quad (14)$$

where $\Omega_m \equiv (8\pi G/3H_0^2)\rho_{m0}$ and $\Omega_\Lambda \equiv \Lambda/3H_0^2$, ρ_{m0} is the critical mass density, Λ is the cosmological constant, and H_0 is the Hubble constant. In the derivation of (14), we have used the conventional Friedmann equation and the definition $H \equiv (da/dt)/a$ in the Λ CDM model as a first-order approximation.

3. NUMERICAL RESULTS FROM SUPERNOVAE

We define a Cartesian coordinate system for the unit 3-vector $\hat{\mathbf{p}}$ corresponding to each SN Ia with the equatorial coordinates (α, δ) , in which $\hat{\mathbf{p}}$ is given as

$$\hat{\mathbf{p}} \equiv \cos(\delta) \cos(\alpha) \hat{\mathbf{i}} + \cos(\delta) \sin(\alpha) \hat{\mathbf{j}} + \sin(\delta) \hat{\mathbf{k}}, \quad (15)$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are basis 3-vectors. Suppose $\hat{\mathbf{n}}$ is given by

$$\hat{\mathbf{n}} \equiv \cos(\delta_0) \cos(\alpha_0) \hat{\mathbf{i}} + \cos(\delta_0) \sin(\alpha_0) \hat{\mathbf{j}} + \sin(\delta_0) \hat{\mathbf{k}}, \quad (16)$$

then the cosine of the angle θ between these two vectors as

$$\cos \theta \equiv \hat{\mathbf{n}} \cdot \hat{\mathbf{p}}. \quad (17)$$

Substituting the relations (8) and (17) into (14), one has

$$\frac{H_0 d_L}{1+z} = \int_0^z \mathcal{B}(z', \theta) \frac{dz'/(1+z')}{\sqrt{\Omega_m \left(\frac{\mathcal{A}}{1+z'}\right)^{-3} + \Omega_\Lambda}}, \quad (18)$$

where

$$\mathcal{A} \equiv 1 - D \cos \theta, \quad (19)$$

$$\mathcal{B}(z', \theta) \equiv \left[\left(\frac{\mathcal{A}}{1+z'} \right)^2 + D^2 \cos^2 \theta \right]^{-1/2}. \quad (20)$$

These relations refer to the anisotropic Hubble diagram of SNe Ia. The direction dependence is obvious in the above equations since different modifications are introduced to the distance-redshift relation of SNe Ia in different spatial directions. Once again, we see that the 1-form in the Randers structure leads to the privileged axis in the universe. In the Λ CDM model, D vanishes. Thus, \mathcal{A} and \mathcal{B} reduce back to 1 and $1+z$, respectively.

We base our numerical study on the Union2.1 dataset [34]. The Union2.1 compilation consists of 580 SNe Ia. Only 389 of them have available equatorial position data in the database of the Central Bureau for Astronomical Telegrams (CBAT) [42]. Thus, we perform a least- χ^2 fit to the data of these SNe Ia to determine the privileged axis (α_0, δ_0) and the geometrical parameter D (viewed as a constant for simplicity). The χ^2 statistic in our fit is

$$\chi^2 \equiv \sum_{i=1}^{389} \frac{[\mu_{\text{th}}(z_i, \alpha_i, \delta_i; \alpha_0, \delta_0, D) - \mu_{\text{obs}}(z_i)]^2}{\sigma_\mu(z_i)^2}, \quad (21)$$

where $\mu_{\text{th}}(z_i, \alpha_i, \delta_i; \alpha_0, \delta_0, D)$ is the distance modulus calculated from (18) by using the definition $\mu \equiv 5 \log_{10}[d_L(\text{Mpc})] + 25$. $\mu_{\text{obs}}(z_i)$ and $\sigma_\mu(z_i)$ respectively denote the observational values of the distance modulus and measurement errors, which are obtained from the Union2.1 compilation. As the first-order approximation, the parameters $\Omega_\Lambda = 0.73$, $\Omega_m = 0.27$, and the Hubble parameter $h = 0.72$ are used in our fit. In the FRW-Randers spacetime, D is around ~ 0.1 , which leads to only small perturbations to the equation (18).

The best-fit results are given as $D = -0.13 \pm 0.05$, and $(\alpha_0, \delta_0) = (68^\circ \pm 44^\circ, -39^\circ \pm 23^\circ)$ in the equatorial coordinate system or $(l_0, b_0) = (242^\circ \pm 44^\circ, -42^\circ \pm 23^\circ)$ in the galactic coordinate system, for a minimum value of $\chi^2_{\text{min}} \simeq 0.99$. All the results are presented in a sense of 68% confidence level (C.L.). The distance modulus vs. redshift relation of the SNe Ia is shown in Fig. 1. This privileged axis obtained by di-

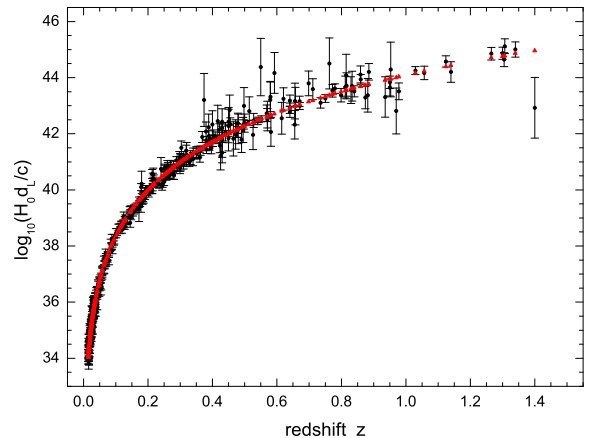


FIG. 1: The distance modulus vs. redshift relation of the SNe Ia in the FRW-Randers spacetime. The (black) dots and errorbars denote experimental data which comes from Union2.1 compilation [34]. The (red) triangles denote theoretical predictions for the SNe Ia in our model.

rect analysis is close to those obtained by comparing the best-fit Hubble diagrams in pairs of hemispheres, see Fig. 2. For example, Antoniou & Perivolaropoulos [10] showed $(l_0, b_0) = (309^\circ_{-3^\circ}^{+23^\circ}, 18^\circ_{-10^\circ}^{+11^\circ})$, Cai & Tuo [26] got $(l_0, b_0) = (314^\circ_{-13^\circ}^{+20^\circ}, 28^\circ_{-33^\circ}^{+11^\circ})$, and Kalus *et al.* [27] obtained $(l_0, b_0) \approx (-35^\circ, -19^\circ)$ (95 % C.L.).

4. CONCLUSIONS AND REMARKS

In this paper, we proposed an anisotropic modification to the FRW line element. The modified line element refers to the Randers spacetime, which possesses the local

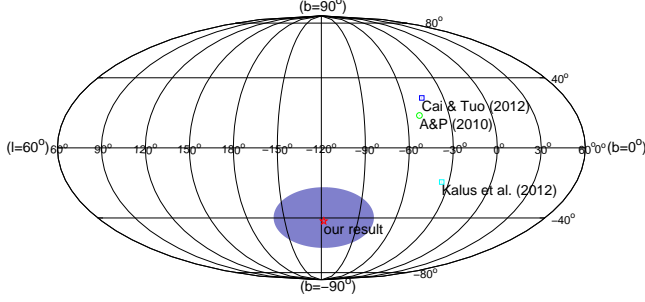


FIG. 2: The privileged axis in the galactic coordinate. The star denotes the direction of the privileged axis obtained by the direct analysis in this paper. For contrast, we also show directions for other privileged axes [10, 26, 27] obtained via the hemisphere comparison method.

symmetry of Cohen & Glashow's VSR. The local symmetry involves the group $TE(2)$. The Euclidean group $E(2)$ contains three generators $T_1 \equiv K_x + J_y$, $T_2 \equiv K_y - J_x$, and J_z , where K_i and J_i ($i = x, y, z$) denote the generators of boosts and rotations, respectively. The former two generators form a two-parameter group of translations in the $x - y$ plane. Thus, the local FRW-Randers spacetime is cylindrically symmetric. However, the parity violates in the z -direction. Otherwise, the $E(2)$ would be enlarged to the Lorentz group [28]. Actually, the 1-form $\tilde{b}_\mu dx^\mu$ in the Randers structure changes its sign under the direction reversal $dx^\mu/d\tau \rightarrow -dx^\mu/d\tau$. This reveals that the Randers structure is asymmetric. On the other hand, the indicatrix of the FRW-Randers spacetime is quadratic, while the center of the indicatrix is displaced in the z -direction [33]. The above two accounts imply a privileged axis in the FRW-Randers spacetime. The existence of the privileged axis contradicts with the cosmological principle and implies a statistically anisotropic universe. We extracted the direction $(l_0, b_0) = (242^\circ \pm 44^\circ, -42^\circ \pm 23^\circ)$ for the privileged axis b^i , based on the Union2.1 compilation of the SNe Ia. The direction is close to those obtained by comparing the best-fit Hubble diagrams in pairs of hemispheres.

It is noteworthy that the Randers spacetime belongs to Finsler geometry [35]. Actually, Finsler geometry gets rid of the quadratic restriction on the spacetime structure (line element) [43]. It is a natural generalization of Riemann geometry and includes Riemann geometry as a special case. Finsler spacetime could depend on certain preferred directions of the spacetime background [29, 44–49]. This could also be revealed via the isometric transformation [30, 50, 51]. There are no more than $\left(\frac{d(d-1)}{2} + 1\right)$ Killing vectors in the d -dimensional Finsler spacetime [50]. Otherwise, Finsler spacetime becomes

Riemannian. Thus, it might be a reasonable candidate to account for the privileged axis and the anisotropic properties of the universe.

We thank useful discussions with Jian-Ping Dai, Yunguo Jiang, Danning Li, and Hai-Nan Lin. This work is supported by the National Natural Science Fund of China under Grant No. 11075166.

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- [1] S. Weinberg, *Cosmology*, Oxford University Press, New York, 2008.
 - [2] E. Komatsu *et al.* [WMAP collaboration], *Astrophys. J. Suppl.* **192**, 18 (2011).
 - [3] B.A. Reid *et al.*, *Mon. Not. Roy. Astron. Soc.* **404**, 60 (2010).
 - [4] L. Perivolaropoulos, arXiv:1104.0539.
 - [5] R. Watkins, H. A. Feldman, and M. J. Hudson, *Mon. Not. R. Astron. Soc.* **392**, 743 (2009).
 - [6] C.L. Bennett *et al.*, *Astrophys. J. Suppl.* **192**, 17 (2011).
 - [7] K. Land and J. Magueijo, *Phys. Rev. Lett.* **95**, 071301 (2005).
 - [8] M. Tegmark, A. de Oliveira-Costa, and A. Hamilton, *Phys. Rev. D* **68**, 123523 (2003).
 - [9] D. Hutsemekers, R. Cabanac, H. Lamy, and D. Sluse, *Astron. Astrophys.* **441**, 915 (2005).
 - [10] I. Antoniou and L. Perivolaropoulos, *JCAP* **1012**, 012 (2010).
 - [11] T.S. Kolatt and O. Lahav, *Mon. Not. Roy. Astron. Soc.* **323**, 859 (2001).
 - [12] C. Bonvin, R. Durrer, and M. Kunz, *Phys. Rev. Lett.* **96**, 191302 (2006).
 - [13] C. Gordon, K. Land, and A. Slosar, *Phys. Rev. Lett.* **99**, 081301 (2007).
 - [14] D.J. Schwarz and B. Weinhorst, *Astron. Astrophys.* **474**, 717 (2007).
 - [15] S. Gupta, T.D. Saini, and T. Laskar, *Mon. Not. Roy. Astron. Soc.* **388**, 242 (2008).
 - [16] T.S. Koivisto and D.F. Mota, *JCAP* **0806**, 018 (2008).
 - [17] M. Blomqvist, J. Enander, and E. Mortsell, *JCAP* **10**, 018 (2010).
 - [18] A. Cooray, D.E. Holz, and R. Caldwell, *JCAP* **11**, 015 (2010).
 - [19] S. Gupta and T.D. Saini, *Mon. Not. Roy. Astron. Soc.* **407**, 651 (2010).
 - [20] T.S. Koivisto, D.F. Mota, M. Quartin, and T.G. Zlosnik, *Phys. Rev. D* **83**, 023509 (2011).
 - [21] J. Colin, R. Mohayaee, S. Sarkar, and A. Shafieloo, *Mon. Not. Roy. Astron. Soc.* **414**, 264 (2011).
 - [22] L. Campanelli, P. Cea, G.L. Fogli, and A. Marrone, *Phys. Rev. D* **83**, 103503 (2011).
 - [23] A.G. Riess *et al.*, *Astron. J.* **116**, 1009 (1998).
 - [24] S. Perlmutter *et al.*, *Astron. J.* **517**, 565 (1999).
 - [25] R. Amanullah *et al.*, *Astrophys. J.* **716**, 712 (2010).
 - [26] R.-G. Cai and Z.-L. Tuo, *JCAP* **1202**, 004 (2012).
 - [27] B. Kalus, D.J. Schwarz, M. Seikel, and A. Wiegand, arXiv:1212.3691.
 - [28] A.G. Cohen and S.L. Glashow, *Phys. Rev. Lett.* **97**, 021601 (2006).
 - [29] G.W. Gibbons, J. Gomis and C.N. Pope, *Phys. Rev. D* **76**, 081701 (2007).

- [30] X. Li and Z. Chang, *Differ. Geom. Appl.* **30**, 737 (2012).
- [31] L. Zhang and X. Xue, arXiv:1205.1134.
- [32] L. Zhang and X. Xue, arXiv:1204.6425.
- [33] G. Randers, *Phys. Rev.* **59**, 195(1941).
- [34] N. Suzuki *et al.*, *Astrophys. J* **746**, 85 (2012).
- [35] D. Bao, S.S. Chern, and Z. Shen, *An Introduction to Riemann–Finsler Geometry*, Graduate Texts in Mathematics **200**, Springer, New York, 2000.
- [36] T. Kahniashvili, G. Lavrelashvili, and B. Ratra, *Phys. Rev. D* **78**, 063012 (2008).
- [37] J.D. Barrow, P.G. Ferreira, and J. Silk, *Phys. Rev. Lett.* **78**, 3610 (1997).
- [38] L. Campanelli, *Phys. Rev. D* **80**, 063006 (2009).
- [39] J. Kim and P. Naselsky, *JCAP* **07**, 041 (2009).
- [40] Z. Chang, S. Wang, and X. Li, *Eur. Phys. J. C* **72**, 1838 (2012).
- [41] S. Dodelson, *Modern Cosmology*, Elsevier (Singapore) Pte Ltd., Singapore, 2008.
- [42] <http://www.cbat.eps.harvard.edu/iau/cbat.html>.
- [43] S.S. Chern, *Notice of AMS*, 959 (1996).
- [44] G.Yu. Bogoslovsky, *Il Nuovo Cimento B* **40**, 99 (1977).
- [45] G.Yu. Bogoslovsky, *Il Nuovo Cimento B* **40**, 116 (1977).
- [46] G.Yu. Bogoslovsky, *Il Nuovo Cimento B* **43**, 377 (1978).
- [47] Z. Chang and S. Wang, *Eur. Phys. J. C* **72**, 2165 (2012).
- [48] Z. Chang and S. Wang, *Eur. Phys. J. C* **73**, 2337 (2013).
- [49] V. Balan, G.Yu. Bogoslovsky, S.S. Kokarev, D.G. Pavlov, S.V. Siparov, and N. Voicu, *J. Mod. Phys.* **3**, 1314 (2012).
- [50] H.C. Wang, *J. London Math. Soc.* **s1-22** (1), 5 (1947).
- [51] S.F. Rutz, *Contemp. Math.* **196**, 289 (1996).